Electromagnetic Theory

Differential Operator, $\vec{\nabla}$

Differential Operator, $\vec{\nabla}$ is defined as $\vec{\nabla}= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Laplacian Operator, ∇^2

Laplacian Operator, ∇^2 is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Gradient of a Scalar Function ($\nabla\phi$)

If $\phi(x, y, z)$ is scalar function of the co-ordinate (x, y, z) be defined and differentiable at each point (x, y, z) in a region of space then the gradient of ϕ ($grad\ \phi$) is dot product of $\vec{\nabla}$ and ϕ

$$ie.\ grad\ \phi = \vec{\nabla} \cdot \phi = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \cdot \phi$$

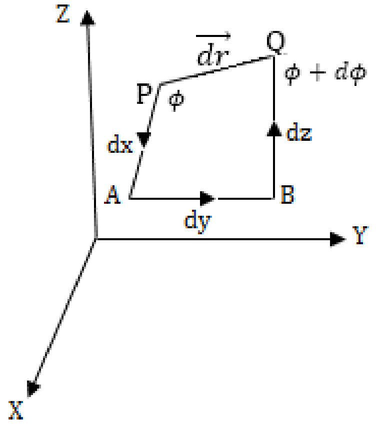
$$grad\ \phi = \vec{\nabla} \cdot \phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

Gradient of scalar function is vector quantity.

Physical Interpretation of Gradient of a Scalar Function

Consider a point P having position co-ordinates x, y, z and a point Q having position co-ordinates $x + dx$, $y + dy$, $z + dz$ in scalar field . The position vector (vector distance) between P and Q is given by

$\overrightarrow{dr} = \hat{\imath}\ dx + \hat{\jmath}\ dy + \hat{k}\ dz$.



Let ϕ be the value of scalar function at P and $\phi + d\phi$ be the value of scalar function at Q.

Then the change in the value of ϕ with respect to distance $\overrightarrow{dr}$ is known as the total differential of ϕ ie $\frac{d\phi}{dr}$

Scalar value at P is ϕ

Scalar value at A = Scalar value at P+(change of scalar function along 'X' direction X dx)

$$ie.\ \text{Scalar value at } A = \phi + \frac{\partial \phi}{\partial x}\ dx$$

Scalar value at B = Scalar value at A +(change of this scalar function along 'Y' direction X dy)

$$ie.\ \text{Scalar value at } B = \phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial}{\partial y}\left(\phi + \frac{\partial \phi}{\partial x}\ dx\right) \cdot dy$$

$$= \phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy + \frac{\partial^2 \phi}{\partial y \partial x}\ dx\ dy$$

$\because$ the term $\frac{\partial^2 \phi}{\partial y \partial x}\ dx\ dy$ is small and hence neglected

$$\therefore \text{Scalar value at } B = \phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy$$

Scalar value at Q = Scalar value at B +(change of this scalar function along 'Z' direction X dz)

$ie.$ Scalar value at Q

$$= \phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy + \frac{\partial}{\partial z}\left(\phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy\right) \cdot dz$$

$$= \phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy + \frac{\partial \phi}{\partial z}\ dz + \frac{\partial^2 \phi}{\partial z \partial x}\ dx\ dz + \frac{\partial^2 \phi}{\partial z \partial y}\ dy\ dz$$

$\because$ the term $\frac{\partial^2 \phi}{\partial z \partial x}\ dx\ dz$ & $\frac{\partial^2 \phi}{\partial z \partial y}\ dy\ dz$ are small and hence neglected

$$\therefore \text{Scalar value at } Q = \phi + \frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy + \frac{\partial \phi}{\partial z}\ dz \ldots\ldots (1)$$

$$\text{Also, Scalar value at } Q = \phi + d\phi \ldots\ldots (2)$$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

From eq^n (1) & (2), we get

$$\therefore \phi + d\phi = \phi + \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y}\, dy + \frac{\partial\phi}{\partial z}\, dz$$

$$\therefore d\phi = \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y}\, dy + \frac{\partial\phi}{\partial z}\, dz \ldots\ldots (3)$$

Now, $\vec{\nabla}\phi \cdot \overrightarrow{dr} = (\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}) \cdot (\hat{\imath}\, dx + \hat{\jmath}\, dy + \hat{k}\, dz\,)$

$$\vec{\nabla}\phi \cdot \overrightarrow{dr} = \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y}\, dy + \frac{\partial\phi}{\partial z}\, dz \ldots\ldots (4)$$

From eq^n (3) & (4), we get

$$\vec{\nabla}\phi \cdot \overrightarrow{dr} = d\phi$$

$$\vec{\nabla}\phi\, dr\, \hat{r} = d\phi$$

$$\vec{\nabla}\phi\ \hat{r} = \frac{d\phi}{dr}$$

Where $\hat{r}$ is unit vector along $PQ(\overrightarrow{dr})$

Hence gradient of scalar function is maximum rate of change of scalar function (ϕ) along the direction which is same as position vector $PQ(\overrightarrow{dr})$.

Physical Significance of Gradient of a scalar function

i) Gradient of scalar function is maximum rate of change of scalar function (ϕ) along the direction which is same as position vector.
ii) If gradient of scalar function is positive along some direction then scalar function increases along that direction.
iii) If gradient of scalar function is negative along some direction then scalar function decreases along that direction.
iv) If gradient of scalar function is zero along some direction then scalar function doesn't change along that direction.

Divergence of a Vector ($\vec{\nabla} \cdot \vec{A}$)

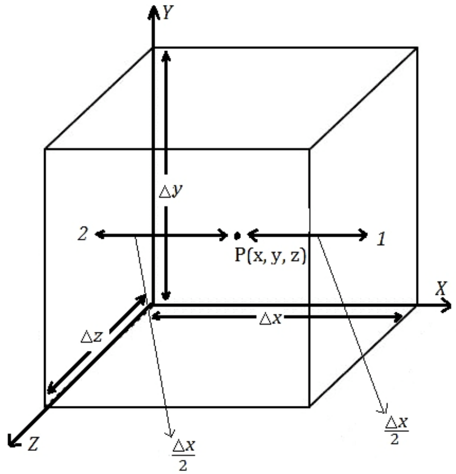
If $\vec{A}(x, y, z) = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ be defined and differentiable at each point (x, y, z) in a region of space then the divergence of $\vec{A}$ is dot product of $\vec{\nabla}$ and $\vec{A}$

$$ie.\, div\, \vec{A} = \vec{\nabla} \cdot \vec{A} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \cdot (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})$$

$$div\, \vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence of vector is scalar quantity.

Physical Interpretation of Divergence of a Vector



Let $\vec{A}$ be the vector at $P(x, y, z)$. Draw a rectangular parallelepiped with P at its centre and edges of length $\Delta x, \Delta y$ & Δz parallel to X, Y & Z axis respectively. A vector $\vec{A}$ at the point P is resolved into A_x, A_y & A_z parallel to three rectangular coordinates.

The value of div$\vec{A}$ is equal to the total outflow of the vector $\vec{A}$ over the six faces of the rectangular parallelepiped per unit volume.

The coordinates the centre of the face 1 are $(x + \frac{\Delta x}{2}$, y, z). The outflow the vector over face 1 and 2 which is perpendicular to X-axis is entirely due to A_x.

Let $\frac{\partial A_x}{\partial x}$ be the rate of the flow of vector along X-axis. Hence flow of vector at the centre of the face 1 is given by,

$$A_x + \frac{\partial A_x}{\partial x} \cdot \frac{\Delta x}{2}$$

$(vector\ at\ P + unit\ change\ in\ vector\ alon\ X\ axis\ X\ distance\ from\ P)$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

The net flow of vector from the face 1 is given by

$(A_x + \frac{\partial A_x}{\partial x} \cdot \frac{\Delta x}{2}) \cdot \Delta y \, \Delta z$

$(vector\ at\ the\ centre\ of\ the\ face\ 1\ X\ Area\ fo\ face\ 1)$

It is in outward direction from P.

Similarly, net flow vectors from the face 2 is given by

$(A_x - \frac{\partial A_x}{\partial x} \cdot \frac{\Delta x}{2}) \cdot \Delta y \, \Delta z$

It is in inward direction from P.

Then the net flow of vectors along X- direction. ie from the face 1 and face 2 is given by

$$\left(A_x + \frac{\partial A_x}{\partial x} \cdot \frac{\Delta x}{2}\right) \cdot \Delta y \, \Delta z - (A_x - \frac{\partial A_x}{\partial x} \cdot \frac{\Delta x}{2}) \cdot \Delta y \, \Delta z$$

$$= 2\frac{\partial A_x}{\partial x} \cdot \frac{\Delta x}{2} \Delta y \, \Delta z$$

$$= \frac{\partial A_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

Similarly, net flow of vectors along Y − direction $= \frac{\partial A_y}{\partial y} \Delta x \, \Delta y \, \Delta z$

& net flow of vectors along Z − direction $= \frac{\partial A_z}{\partial z} \Delta x \, \Delta y \, \Delta z$

$$\therefore \text{ Net flow of vectors} = \frac{\partial A_x}{\partial x} \Delta x \, \Delta y \, \Delta z + \frac{\partial A_y}{\partial y} \Delta x \, \Delta y \, \Delta z + \frac{\partial A_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

$$= (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}) \Delta x \, \Delta y \, \Delta z$$

$$\therefore \text{ Net flow of vectors per unit volume} = (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}) \frac{\Delta x \, \Delta y \, \Delta z}{\Delta x \, \Delta y \, \Delta z}$$

(Where $\Delta x \, \Delta y \, \Delta z$ is volume of rectangular parallelepiped)

$$\therefore \text{ Net flow of vectors per unit volume} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= div\, \vec{A} = \vec{\nabla} \cdot \vec{A}$$

Hence divergence of vector is net outflow of vectors (net flux) per unit volume.

Physical Significance of Divergence of a Vector

i) Divergence of vector is net outflow of vectors (net flux) per unit volume.

ii) If divergence at a point is positive then flux expands (leaves) from that point or that point is source of that vector.

iii) If divergence at a point is negative then flux contracts (enters) to that point or that point is sink of that vector.

iv) If the divergence is zero at a point then amount of flux entering is equal to amount of flux leaving.

v) For small volume, concentration of vector field lines is maximum and vice-versa. Hence divergence (for the same vector field) is maximum for near to source (small volume) and vice-versa.

Curl of a Vector ($\vec{\nabla}\ X\ \vec{A}$)

If $\vec{A}(x, y, z) = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ be defined and differentiable at each point (x, y, z) in a region of space then the curl of $\vec{A}$ is cross product of $\vec{\nabla}$ and $\vec{A}$

$$ie.\, curl\, \vec{A} = \vec{\nabla}\ X\ \vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) X (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})$$

$$\vec{\nabla}\ X\ \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

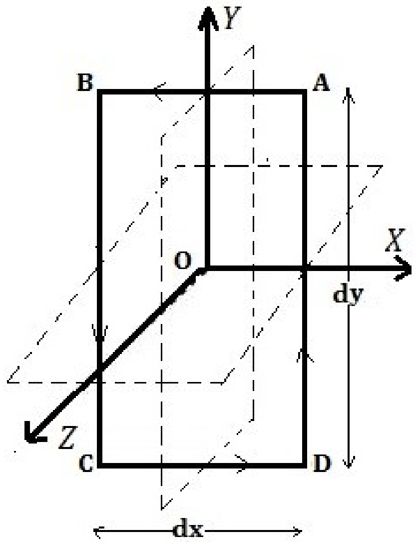
$$\vec{\nabla}\ X\ \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Curl of a vector is vector quantity.

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

Physical Interpretation of Curl of a Vector

Consider three small rectangular areas intersecting mutually at right angles to each other at centre O where the vector is $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.



Consider a rectangular ABCD with sides dx and dy and its face normal to Z-axis. 'O' lies at the centre of ABCD

Since A_x, A_y & A_z are the vector co-ordinates of $\vec{A}$ at 'O', the value of vector at the centres of AB, BC, CD and DA are

$A_x + \frac{\partial A_x}{\partial y}\cdot\frac{dy}{2}, A_y - \frac{\partial A_y}{\partial x}\cdot\frac{dy}{2}, A_x - \frac{\partial A_x}{\partial y}\cdot\frac{dy}{2}$ & $A_y + \frac{\partial A_y}{\partial x}\cdot\frac{dy}{2}$ respectively

Line integral of $\vec{A}$ over ABCD (in Clockwise direction)

$$= \left(A_x + \frac{\partial A_x}{\partial y}\cdot\frac{dy}{2}\right)(-dx) + \left(A_y - \frac{\partial A_y}{\partial x}\cdot\frac{dy}{2}\right)(-dy) + \left(A_x - \frac{\partial A_x}{\partial y}\cdot\frac{dy}{2}\right)(dx) + \left(A_y + \frac{\partial A_y}{\partial x}\cdot\frac{dy}{2}\right)(dy)$$

$$= \left[\left(A_x - \frac{\partial A_x}{\partial y}\cdot\frac{dy}{2}\right) - \left(A_x + \frac{\partial A_x}{\partial y}\cdot\frac{dy}{2}\right)\right]dx + \left[\left(A_y + \frac{\partial A_y}{\partial x}\cdot\frac{dy}{2}\right) - \left(A_y - \frac{\partial A_y}{\partial x}\cdot\frac{dy}{2}\right)\right]dy$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dx\,dy$$

∴ Line integral of $\vec{A}$ over ABCD per unit area

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\frac{dx\,dy}{dx\,dy}$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Line integral of $\vec{A}$ over ABCD per unit area is Z component of $curl\,\vec{A}$

$$(curl\,\vec{A})_Z = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

Similarly X & Y components of $curl\,\vec{A}$ are

$$(curl\,\vec{A})_X = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} \;\&\; (curl\,\vec{A})_Y = \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\jmath}$$

$$\therefore Curl\,\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

Hence curl of a vector over a closed path is net line integral of a vector over that closed path per unit its area.

Physical Significance of curl of a Vector

i) Curl of a vector is net line integral of a vector per unit area.
ii) Curl of a vector measures the rotation of vector field.
iii) If curl of a vector is non zero then the vector field has rotation.
iv) If curl of a vector is zero then vector field has no rotation and called as conservative field.

Gauss divergence theorem

Statement: The surface integral of the normal component of a vector ($\vec{A}$) taken over closed surface (S) is equal to the volume integral of the divergence of same vector ($\vec{A}$) taken over closed volume (V) enclosed by same surface (S).

$$ie. \iint_S \vec{A}\;ds = \iiint_V (div\,\vec{A})\,dv$$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

Stokes theorem

Statement: The line integral of a vector $(\vec{A})$ taken around a closed curve (C) is equal to the surface integral of the curl of the same vector $(\vec{A})$ taken over surface(S) bounded by same closed path (C).

$$\oint_l \vec{A}\, dr = \iint_S curl\, \vec{A}\;\; ds$$

Maxwell's Equations

There are four fundamental equations of electromagnetism known as Maxwell's equations which represented in differential form as,

1) $\vec{\nabla} \cdot \vec{D} = \rho$

It represents Gauss law in electrostatics. Where $\vec{D}$ is electric displacement and ρ is the free charge density.

2) $\vec{\nabla} \cdot \vec{B} = 0$

It represents Gauss law in magnetostatics. Where $\vec{B}$ is magnetic induction.

3) $\vec{\nabla}\, X\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

It represents Faraday's law of electromagnetic induction. Where $\vec{E}$ is electric field intensity and $\vec{B}$ is magnetic induction.

4) $\vec{\nabla}\, X\, \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

It represents Maxwell's modification in Amperes's law. Where $\vec{E}$ is electric field intensity and $\vec{B}$ is magnetic induction.

Derivations of Maxwell's Equations

1) $div\, \vec{D} = \vec{\nabla} \cdot \vec{D} = \rho$

Consider a surface S bounding a volume V in dielectric medium under electric field. So we have two types of charges free charges and induced charges.

Let ρ and ρ' be the charge densities of free charges and induced charges.

According to Gauss law in electrostatics
Electric flux linked with closed surface,

$$\phi_E = \frac{1}{\varepsilon_0} q \quad \text{(where q is net charge present on surface S)}$$

$$\because\ \phi_E = \int_s \vec{E}\, ds\ \ \&\ q = \int_v (\rho + \rho')dv$$

$\therefore$ Above equation becomes

$$\int_s \vec{E}\, ds = \frac{1}{\varepsilon_0}\int_v (\rho + \rho')dv$$

$$\varepsilon_0 \int_s \vec{E}\, ds = \int_v \rho\, dv + \int_v \rho' dv$$

$$\because\ \rho' = -div\, \vec{P}$$

$\therefore$ Above equation becomes

$$\int_s \varepsilon_0 \vec{E}\, ds = \int_v \rho\, dv - \int_v div\, \vec{P}\, dv$$

Using Gauss Divergence theorem on left side to change surface integral to volume integral, we get

$$\int_v div\, (\varepsilon_0 \vec{E})\, dv = \int_v \rho\, dv - \int_v div\, \vec{P}\, dv$$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

$$\int_v div\,(\varepsilon_0\vec{E}+\vec{P})\,dv=\int_v \rho\,dv$$

$\because\ \varepsilon_0\vec{E}+\vec{P}=\vec{D}$, Electric Displacement

$\therefore$ Above equation becomes

$$\int_v div\,\vec{D}\,dv=\int_v \rho\,dv$$

$$\int_v (div\,\vec{D}-\rho)\,dv=0$$

Since this equation is true for all volume, this integral vanishes

$$\therefore\ div\,\vec{D}-\rho=0$$

$$div\,\vec{D}=\rho \quad \text{or} \quad \vec{\nabla}\cdot\vec{D}=\rho$$

2) $\vec{\nabla}\cdot\vec{B}=0$

Experimental results shows that the number of magnetic field lines entering to any arbitrary closed surface is equal to number of magnetic field lines leaving from it. Therefore the flux of magnetic induction $\vec{B}$ across any closed surface is always zero. ie.

$$\int_s \vec{B}\,ds=0$$

Using Gauss Divergence theorem on left side to change surface integral to volume integral, we get

$$\int_v div\,\vec{B}\,dv=0$$

Since this equation is true for all volume, this integral vanishes

$$div\,\vec{B}=0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{B}=0$$

3) $\vec{\nabla}\,X\,\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

Faraday's law states that induced emf in a circuit is proportional to negative rate of change of magnetic flux linked with it. According the Faraday's law,

Induced emf, $e=-\dfrac{d\phi_E}{dt}$

But, magnetic flux, $\phi_E=\int_s \vec{B}\,ds$

Where S is surface of closed loop C

$$\therefore\ e=-\frac{d}{dt}\left(\int_s \vec{B}\,ds\right)$$

$$=-\left(\int_s \frac{\partial\vec{B}}{\partial t}\,ds \qquad \ldots\ldots\ldots(1)\right.$$

We know emf is work done in carrying a unit charge around a closed loop in an electric field. Therefore,

$$e=\int_c \vec{E}\,dl \qquad \ldots\ldots\ldots(2)$$

Comparing equation (1) & (2), we get

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

$$\int_c \vec{E}\,dl = -\int_s \frac{\partial \vec{B}}{\partial t}\,ds$$

Using Stokes theorem to change line integral into surface integral, we get,

$$\int_s curl\,\vec{E}\,ds = -(\int_s \frac{\partial \vec{B}}{\partial t}\,ds$$

$$\int_s \left(curl\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right) ds = 0$$

Since this equation is true for all surface, this integral vanishes

$$\therefore curl\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$curl\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad or \quad \vec{\nabla}\,X\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

4) $\vec{\nabla}\,\boldsymbol{X}\,\vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}}$

Ampere's Circuital law states that the line integral of magnetic field $\vec{H}$ around any closed path or circuit is equal to the current enclosed by the path.

According to Ampere's law, $\int_c \vec{H}\,dl = I$

$\because I = \int_s \vec{J}\,ds$, Where J is current density

The above equation becomes

$$\int_c \vec{H}\,dl = \int_s \vec{J}\,ds$$

Where S is surface bounded by closed path C.

Using Stokes theorem to change line integral into surface integral, we get,

$$\int_s curl\,\vec{H}\,ds = \int_s \vec{J}\,ds$$

$$\int_s \left(curl\,\vec{H} - \vec{J}\right) ds = 0$$

Since this equation is true for all surface, this integral vanishes

$$\therefore curl\,\vec{H} - \vec{J} = 0$$

$$curl\,\vec{H} = \vec{J} \qquad \ldots\ldots\ldots(1)$$

But Maxwell found it to be incomplete for time varying electric field and assumed that a quantity called displacement current and corresponding displacement current density must be included.

$$ie.\ \overrightarrow{J_D} = \frac{\partial \vec{D}}{\partial t}$$

Therefore in equation (1), $\vec{J}$ must be replaced by $\vec{J} + \overrightarrow{J_D}$, therefore equation (1) becomes,

$$curl\,\vec{H} = \vec{J} + \overrightarrow{J_D}$$

$$curl\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

Integral form of Maxwell's equations and their physical significance

1) $\int_s \vec{D}\, ds = q$

This Maxwell's equation signifies
i) The total flux of the electric displacement linked with the closed surface is equal to the total charge enclosed by the close surface.
ii) The positive charges becomes the source of flux of the electric displacement and the negative charges becomes the sink to flux of the electric displacement. Thereby this equation confirms the existence of positive and negative charges.

2) $\int_s \vec{B}\, ds = 0$

This Maxwell's equation signifies
i) The total outward flux of magnetic induction linked with a closed surface is zero. That is magnetic flux lines are continuous forming closed loops implying that there are no sink or sources in the case of magnetic flux.
ii) It confirms the non existence of magnetic monopole and hence establishes that the magnetic poles always appear as dipoles.

3) $\int_c \vec{E}\, dl = -\int_s \frac{\partial \vec{B}}{\partial t}\, ds$

This Maxwell's equation signifies;
Line integral of electric intensity around closed path ie. electromotive force is equal to the negative rate of change of magnetic flux linked with the path.

4) $\int_c \vec{H}\, dl = \int_s \left(J + \frac{\partial \vec{D}}{\partial t}\right) ds$

This Maxwell's equation signifies;
Line integral of magnetic field intensity around closed path ie. magnetomotive force is equal sum of the conduction current and the displacement current linked with that path.

Derivation of general plane wave equation in free space

In the free space, the charge density $\rho = 0$, conductivity $\sigma = 0$ and hence current density $J = \sigma E = 0$. Also $\varepsilon_r = \mu_r = 1$, therefore $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$
Hence Maxwell's equations for free space can be reduced to

$$\vec{\nabla} \cdot \vec{D} = \rho \quad ; \quad \vec{\nabla} \cdot \vec{E} = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad ; \quad \vec{\nabla} \cdot \vec{H} = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\vec{\nabla}\, X\, \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ldots\ldots\ldots\ldots\ldots (3)$$

$$\vec{\nabla}\, X\, \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla}\, X\, \vec{H} = 0 + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \ldots\ldots\ldots\ldots\ldots (4)$$

$$\vec{\nabla}\, X\, \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ldots\ldots\ldots\ldots\ldots (4)$$

From equation (3),

$$\vec{\nabla}\, X(\vec{\nabla}\, X\, \vec{E}) = \vec{\nabla}\, X\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\vec{\nabla}\, X\left(\frac{\partial \vec{B}}{\partial t}\right)$$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

$$= -\frac{\partial}{\partial t}(\vec{\nabla}\, X\, \vec{B})$$

Substitution $\vec{\nabla}\, X\, \vec{B}$ from eq^n (4), we get

$$\vec{\nabla}\, X(\vec{\nabla}\, X\, \vec{E}) = -\frac{\partial}{\partial t}\left(\varepsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\varepsilon_0\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \ldots\ldots\ldots (5)$$

We have the vector identity,

$$\vec{\nabla}\, X(\vec{\nabla}\, X\, \vec{E}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$$

$$= -\nabla^2\vec{E} \ldots\ldots (6) \qquad \because \vec{\nabla}\cdot\vec{E} = 0$$

From eq^n (5) & (6), we get

$$-\nabla^2\vec{E} = -\varepsilon_0\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2\vec{E} = \varepsilon_0\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$\because$ velocity of light, $C = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$; $\varepsilon_0\mu_0 = \frac{1}{C^2}$

Substituting in the above equation, we get

$$\nabla^2\vec{E} = \frac{1}{C^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Equation (7) represents equation of motion of electric wave travelling with the speed of light.

From equation (4),

$$\vec{\nabla}\, X(\vec{\nabla}\, X\, \vec{B}) = \vec{\nabla}\, X\left(\varepsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$= \varepsilon_0\mu_0\, \vec{\nabla}\, X\left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$= \varepsilon_0\mu_0 \frac{\partial}{\partial t}(\vec{\nabla}\, X\, \vec{E})$$

Substitution $\vec{\nabla}\, X\, \vec{E}$ from eq^n (3), we get

$$\vec{\nabla}\, X(\vec{\nabla}\, X\, \vec{B}) = \varepsilon_0\mu_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\varepsilon_0\mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} \ldots\ldots\ldots (8)$$

We have the vector identity,

$$\vec{\nabla}\, X(\vec{\nabla}\, X\, \vec{B}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$$

$$= -\nabla^2\vec{B} \ldots\ldots (9) \qquad \because \vec{\nabla}\cdot\vec{B} = 0$$

From eq^n (8) & (9), we get

$$-\nabla^2\vec{B} = -\varepsilon_0\mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2\vec{B} = \varepsilon_0\mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2\vec{B} = \frac{1}{C^2}\frac{\partial^2 \vec{B}}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10) \quad \because \varepsilon_0\mu_0 = \frac{1}{C^2}$$

Equation (10) represents equation of motion of magnetic wave travelling with the speed of light.

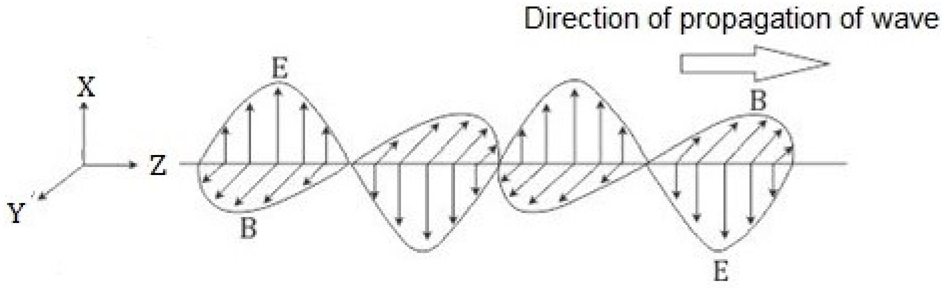
The equation (7) & (10) reveals that these identical in the form to the equation.

$$\nabla^2\vec{\psi} = \frac{1}{V^2}\frac{\partial^2 \vec{\psi}}{\partial t^2} \quad \ldots\ldots\ldots\ldots (11)$$

Equation (11) represents equation of motion of any wave with field intensity ψ and travelling with the speed V.

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

Transverse Nature of Electromagnetic Waves (Radiation)



Consider an electromagnetic wave propagating in free space along Z-direction. Electric field, $\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$ and Magentic field, $\vec{H} = H_x\hat{\imath} + H_y\hat{\jmath} + H_z\hat{k}$ vary only in Z-direction.

$$\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \;\;\&\;\; \frac{\partial}{\partial z} \neq 0 \quad \ldots..(2)$$

Such wave is said to be plane wave since its vectors are functions of(z, t). Hence we can write,

$$\vec{E} = \vec{E}(z,t) \quad \& \quad \vec{H} = \vec{H}(z,t)$$

Using Maxwell's first equation

$$\vec{\nabla}\cdot\vec{D} = \rho = 0 \text{ (free space)}$$

$$\varepsilon_0\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\frac{\partial E_z}{\partial z} = 0 \qquad \text{(from eq}^{\text{n}}\text{(1))}$$

$$\therefore E_z = constant\ wrt.Z \quad \ldots\ldots.(3)$$

Using Maxwell's fourth equation

$$\vec{\nabla}\,X\,\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\,X\,\vec{H} = 0 + \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\,X\,\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Taking Z component of curl

$$\left(\vec{\nabla}\,X\,\vec{H}\right)_Z = \varepsilon_0\frac{\partial E_z}{\partial t}$$

$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{k} = \varepsilon_0\frac{\partial E_z}{\partial t}$$

$$0 = \varepsilon_0\frac{\partial E_z}{\partial t} \qquad \text{(from eq}^{\text{n}}\text{(1))}$$

$$\therefore \frac{\partial E_z}{\partial t} = 0$$

$$\therefore E_z = constant\ wrt.time, t \quad \ldots\ldots.(4)$$

Using Maxwell's second equation

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\mu_0\vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

$$\frac{\partial H_z}{\partial z} = 0 \qquad \text{(from eq}^{\text{n}}\text{(1))}$$

$$\therefore H_z = constant\ wrt.Z \quad \ldots\ldots.(5)$$

Using Maxwell's third equation

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

$$\vec{\nabla} X \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} X \vec{E} = \mu_0 \frac{\partial \vec{H}}{\partial t}$$

Taking Z component of curl

$$\left(\vec{\nabla} X \vec{E}\right)_Z = \mu_0 \frac{\partial H_z}{\partial t}$$

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k} = \mu_0 \frac{\partial H_z}{\partial t}$$

$$0 = \mu_0 \frac{\partial H_z}{\partial t} \qquad (\text{from eq}^{\text{n}}(1))$$

$$\therefore \frac{\partial H_z}{\partial t} = 0$$

$$\therefore H_z = constant\ wrt.time, t\ \ldots\ldots(6)$$

From eq^n (3) & (4) E_z is constant wrt. to space (Z) and time (t) thus it represents static constant. Similarly, from eq^n (5) & (6) H_z is constant wrt. to space (Z) and time (t) thus it represents static constant.

$$\therefore E_z = H_z = 0 \ldots\ldots\ldots\ldots\ldots(7)$$

Since electric wave lies in X-Z plane and magnetic wave lies in Y-Z pane

$$E_y = H_x = 0 \ldots\ldots(8)$$

We have, $\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$

$$\vec{E} = E_x\hat{\imath} + 0 + 0$$

$$\vec{E} = E_x\hat{\imath} \ \ldots\ldots\ldots(9)$$

We have, $\vec{H} = H_x\hat{\imath} + H_y\hat{\jmath} + H_z\hat{k}$

$$\vec{H} = 0 + H_y\hat{\jmath} + 0$$

$$\vec{H} = H_y\hat{\jmath} \ \ldots\ldots\ldots(10)$$

Equation (9) represents electric wave vibrate just along X-direction ie. perpendicular to direction of propagation. Similarly equation (10) represents magnetic wave vibrate just along Y-direction ie. perpendicular to direction of propagation. Hence in electromagnetic wave both electric and magnetic wave are transverse in nature.

Poynting Theorem and Poynting Vector

Statement: In a plane electromagnetic waves, the rate of flow of energy per unit area is proportional the cross product of electric and magnetic field intensities.

Derivation for Poynting Vector

Electromagnetic wave carries energy when it propagates. Let W be the electromagnetic energy per unit area in a medium having permeability μ and permittivity ε is given by

$$W = \frac{1}{2}\left(\mu\vec{H}^2 + \varepsilon\vec{E}^2\right)$$

$$\because \mu\vec{H} = \vec{B} \quad \& \quad \varepsilon\vec{E} = \vec{D}$$

$$\therefore W = \frac{1}{2}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\ldots\ldots\ldots\ldots(1)$$

Let dv be the small volume enclosed by surface ds. The rate of decrease of energy is given by

$$\frac{dW}{dt} = -\frac{d}{dt}\left[\int_v \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right) dv\right]$$

$$= -\frac{1}{2}\int_v \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) dv$$

$$\because \vec{\nabla} X \vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \& \qquad curl\,\vec{H} = \frac{\partial\vec{D}}{\partial t} \ \ (J = 0)$$

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum

$$\therefore \quad = -\frac{1}{2}\int_v \left[\vec{E}\cdot(\vec{\nabla}\,X\,\vec{H}) - \vec{H}\cdot(\vec{\nabla}\,X\,\vec{E})\right] dv \ \ldots\ldots..(2)$$

We know, $\vec{A}\ curl\ \vec{B} - \vec{B}\ curl\ \vec{A} = div\left(\vec{B}\,X\,\vec{A}\right) = -div\left(\vec{A}\,X\,\vec{B}\right)$

Therefore eqn (2) becomes

$$\frac{dW}{dt} = -\frac{1}{2}\int_v \left[-div\,(\vec{E}\,X\,\vec{H})\right] dv$$

$$= \frac{1}{2}\int_v \left[div\,(\vec{E}\,X\,\vec{H})\right] dv$$

Using Gauss Divergence theorem to change volume integral into surface integral, we get,

$$\frac{dW}{dt} = \int_s (\vec{E}\,X\,\vec{H})\,ds \quad \ldots\ldots\ldots\ldots..(3)$$

This gives Poynting Theorem. In this vector $\vec{E}\,X\,\vec{H}$ is called as poynting vector and denoted by $\vec{P}$ and it along the direction of propagation.

$$\therefore \vec{P} = \vec{E}\,X\,\vec{H} \ \ldots\ldots\ldots\ldots..(4)$$

Physical significance

1) $\vec{P}$ represents an amount of energy of electromagnetic wave per unit area per unit time which is along the direction of propagation of electromagnetic wave.
2) Though $\vec{E}$ & $\vec{H}$ being oscillatory, change in direction, $\vec{P}$ remains unidirectional.
3) The magnitude of $\vec{P}$ varies. It is maximum when $\vec{E}$ & $\vec{H}$ are maximum and zeor when $\vec{E}$ & $\vec{H}$ are zero.

Prof. Avadhut Surekha Prakash Manage
D. M. S. Mandal's Bhaurao Kakatkar College, Belgaum